Some notes about matrices, 3

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1 Projective spaces

Let n be a positive integer. As usual, $\mathbf{R}$, $\mathbf{C}$ denote the real and complex numbers, respectively, and $\mathbf{R}^n$, $\mathbf{C}^n$ consist of the n-tuples of real and complex numbers, respectively. The n-dimensional *real and complex projective spaces* $\mathbf{RP}^n$, $\mathbf{CP}^n$ consist of the real and complex lines through the origin in $\mathbf{R}^{n+1}$, $\mathbf{C}^{n+1}$, respectively.

To put it another way, if $\mathbf{R}_*$, $\mathbf{C}_*$ denote the nonzero real and complex numbers, respectively, then we have natural actions of $\mathbf{R}_*$, $\mathbf{C}_*$ on $\mathbf{R}^{n+1}\backslash\{0\}$, $\mathbf{C}^{n+1}\backslash\{0\}$ by scalar multiplication, and the projective spaces are the corresponding quotient spaces. In other words, two nonzero vectors v, w in $\mathbf{R}^{n+1}$, $\mathbf{C}^{n+1}$ lead to the same point in the corresponding projective space exactly when they are scalar multiples of each other. Note that we get canonical mappings from $\mathbf{R}^{n+1}\backslash\{0\}$, $\mathbf{C}^{n+1}\backslash\{0\}$ onto $\mathbf{RP}^n$, $\mathbf{CP}^n$, in which a nonzero vector v is sent to the line through the origin which passes through v, which consists of all scalar multiples of v.

If L is a nontrivial linear subspace of $\mathbf{R}^{n+1}$, $\mathbf{C}^{n+1}$ of dimension $l+1$, say, then we get an interesting space $\mathbf{P}(L)$ consisting of all lines through the origin in L, which we can think of as sitting inside of $\mathbf{RP}^n$, $\mathbf{CP}^n$, as appropriate. More precisely, $\mathbf{P}(L)$ is basically a copy of $\mathbf{RP}^l$ or $\mathbf{CP}^l$. These are the l-dimensional "linear subspaces" of projective space, analogous to linear subspaces of $\mathbf{R}^n$, $\mathbf{C}^n$.

If A is an invertible linear transformation on $\mathbf{R}^{n+1}$ or on $\mathbf{C}^{n+1}$, then A takes lines to lines, and induces a transformation $\widehat{A}$ on the corresponding projective space. Notice that $\widehat{A}$ is automatically a one-to-one transformation of the corresponding projective space onto itself, with

$$\widehat{A}^{-1} = (\widehat{A^{-1}}), \tag{1.1}$$

and $\widehat{A}$ maps linear subspaces of projective space to themselves, in the sense of the preceding paragraph. Also, if A_1, A_2 are invertible linear transformations on $\mathbf{R}^{n+1}$ or on $\mathbf{C}^{n+1}$, then the induced transformations $\widehat{A}_1$, $\widehat{A}_2$ on the corresponding projective space satisfy

$$\widehat{A}_1 \circ \widehat{A}_2 = (\widehat{A_1 \circ A_2}). \tag{1.2}$$

Let H be a hyperplane in $\mathbf{R}^{n+1}$ or in $\mathbf{C}^{n+1}$, which is to say a linear subspace of dimension n, and let v be a nonzero vector in $\mathbf{R}^{n+1}$, $\mathbf{C}^{n+1}$, as appropriate. This leads to an affine hyperplane $H+v$, consisting of all vectors of the form $w+v$, $w \in H$, and which does not contain the vector 0. For each $w \in H$, we can look at the line through $w+v$, which we can view as an element of the corresponding projective space.

In other words, we basically get an embedding of H into the corresponding projective space, $\mathbf{RP}^n$ or $\mathbf{CP}^n$. Of course we can also think of H as being isomorphic to $\mathbf{R}^n$ or $\mathbf{C}^n$, so that we are really looking at a bunch of embeddings of $\mathbf{R}^n$, $\mathbf{C}^n$ into $\mathbf{RP}^n$, $\mathbf{CP}^n$, respectively. For instance, we can do this with H equal to the jth coordinate hyperplane in $\mathbf{R}^{n+1}$, $\mathbf{C}^{n+1}$, $1 \le j \le n+1$, which is defined by the condition that the jth coordinate of vectors in H are equal to 0, and we can take v to be the jth standard basis vector, with jth coordinate equal to 1 and the other n coordinates equal to 0.

These $n+1$ embeddings of $\mathbf{R}^n$, $\mathbf{C}^n$ into $\mathbf{RP}^n$, $\mathbf{CP}^n$ corresponding to the $n+1$ coordinate hyperplanes in $\mathbf{R}^{n+1}$, $\mathbf{CP}^{n+1}$ are sufficient to cover the projective space, i.e., every point in projective space shows up in the image of at least one of the embeddings. For a given hyperplane H, the set of points in the projective space which do not occur in the embedding of H is

the same as $\mathbf{P}(H)$. Thus the set of missing points in the projective space lie in a projective subspace of dimension 1 less.

Using these embeddings of $\mathbf{R}^n$, $\mathbf{C}^n$ into the corresponding projective spaces, we can think of the projective spaces as being manifolds. That is, these embeddings provide local coordinates for all points in the projective space. Two different embeddings which contain the same point p in their image are compatible in terms of topology and also smooth structure, and indeed there is a finer "projective" structure which is reflected in the presence of nice projective subspaces, for instance.

Note that two invertible linear transformations A_1, A_2 on $\mathbf{R}^{n+1}$ or on $\mathbf{C}^{n+1}$ lead to the same induced transformation on projective space if and only if there is a nonzero scalar α such that $A_2 = \alpha\, A_1$. Thus the group of these "projective linear transformations" has dimension $(n+1)^2 - 1$ over the real or complex numbers, as appropriate. Also, for any pair of points p, q in a projective space, there is a projective linear transformation which takes p to q.

2 Grassmannians

Fix positive integers k, n with $k < n$. The Grassmann spaces $G_{\mathbf{R}}(k,n)$, $G_{\mathbf{C}}(k,n)$ consist of the k-dimensional linear subspaces of $\mathbf{R}^n$, $\mathbf{C}^n$, respectively. When $k = 1$ this reduces to the $(n-1)$-dimensional projective spaces.

Suppose that L, M are linear subspaces of $\mathbf{R}^n$ or of $\mathbf{C}^n$ such that L has dimension k, M has dimension $n-k$, and the intersection of L, M consists of only the zero vector. Thus L and M are complementary, and if A is a linear mapping from L to M, then the graph of A, consisting of the vectors

$$v + A(v), \quad v \in L, \tag{2.1}$$

is also a k dimensional subspace of $\mathbf{R}^n$ or of $\mathbf{C}^n$, as appropriate. In this way we can embed the vector space of linear transformations from L to M into the Grassmannian, and in particular this provides a nice coordinate patch around L itself.

In particular, notice that the dimension of the Grassmann space of k planes in $\mathbf{R}^n$ or $\mathbf{C}^n$ is

$$k(n-k) \tag{2.2}$$

with respect to the real or complex numbers, as appropriate. Just as for projective spaces, invertible linear transformations on $\mathbf{R}^n$ or on $\mathbf{C}^n$ induce

interesting mappings on the corresponding Grassmannians. These actions are again transitive, because if L_1, L_2 are k-dimensional linear subspaces of $\mathbf{R}^n$ or of $\mathbf{C}^n$, then there is an invertible linear transformation A on $\mathbf{R}^n$ or on $\mathbf{C}^n$, as appropriate, such that $A(L_1) = L_2$.

There is a natural correspondence between the Grassmann spaces of k-dimensional linear subspaces in $\mathbf{R}^n$ or in $\mathbf{C}^n$ and the Grassmann spaces of $(n-k)$-dimensional linear subspaces of $\mathbf{R}^n$ or in $\mathbf{C}^n$, respectively. One might prefer to think of k-dimensional linear subspaces of $\mathbf{R}^n$ or $\mathbf{C}^n$ as being associated to $(n-k)$-dimensional linear subspaces of the corresponding dual spaces, by looking at intersections of kernels of linear functionals in a subspace of a dual space. Alternatively, one can think of linear subspaces of dimension k as being associated to their orthogonal complements, which are linear subspaces of dimension $n-k$, using inner products.

3 Hopf-type spaces

Fix a positive integer n, and consider $\mathbf{R}^n \backslash \{0\}$, $\mathbf{C}^n \backslash \{0\}$. On these spaces one can consider multiplication by integer powers of 2. One can then consider the corresponding quotients by this action, which is to say to identify two nonzero vectors which can be expressed as integer powers of 2 times each other.

There are natural mappings from these spaces to the corresponding projective spaces of dimension $n-1$, by looking at the lines through nonzero vectors, which are not changed by nonzero scalar multiples in general. As for projective spaces, invertible linear mappings on $\mathbf{R}^n$ or on $\mathbf{C}^n$ induce nice transformations on the quotients by the actions of powers of 2, and indeed these transformations are compatible with the ones on projective spaces too. In the quotients by integer powers of 2, there are interesting subspaces arising from linear subspaces back in $\mathbf{R}^n$ or in $\mathbf{C}^n$.

Of course one can consider integer powers of other scalars with absolute value larger than 1 in place of 2, and have similar properties. Analogous objects have been studied using more complicated invertible linear transformations on $\mathbf{R}^n$ or on $\mathbf{C}^n$ rather than scalar multiplication. This can lead to more tricky kinds of twisting, for instance.

4 Hopf fibrations

Let n be a positive integer. There is an obvious mapping from the unit sphere in $\mathbf{R}^{n+1}$, consisting of vectors with norm 1, to n-dimensional real projective space $\mathbf{RP}^n$. Namely, if $x \in \mathbf{R}^{n+1}$ and $|x| = 1$, the line through x leads to a point in $\mathbf{RP}^n$, and every element of $\mathbf{RP}^n$ arises in this manner exactly twice.

Similarly, there is a natural mapping from the unit sphere in $\mathbf{C}^{n+1}$, consisting of vectors with norm 1, to complex projective space $\mathbf{CP}^n$. If v is a vector in $\mathbf{C}^{n+1}$ with $|v| = 1$, then the line through v leads to a point in $\mathbf{C}^n$. Two vectors v, w in $\mathbf{C}^n$ with $|v| = |w| = 1$ correspond to the same element of $\mathbf{CP}^n$ if and only if there is a complex number α such that $|\alpha| = 1$ and $w = \alpha\, v$.

In other words, the inverse image of a point in $\mathbf{RP}^n$ back in the unit sphere of $\mathbf{R}^{n+1}$ consists of two points, while in the complex case the inverse image of a point in $\mathbf{CP}^n$ back in the unit sphere in $\mathbf{C}^{n+1}$ is a circle. In terms of the unit sphere in $\mathbf{C}^{n+1}$, there is a family of circles which cover the whole sphere and which are pairwise disjoint. When $n = 1$, these are circles in the 3-dimensional sphere which are linked with each other.

The $n = 1$ case has other nice properties. One can identify $\mathbf{CP}^1$ with a 2-dimensional sphere, basically because $\mathbf{CP}^1$ can be expressed as $\mathbf{C}$ with a single additional point, which one might denote ∞. With respect to this identification, projective linear transformations on $\mathbf{CP}^1$ correspond to transformations of the form

$$z \mapsto \frac{a\, z + b}{c\, z + d} \tag{4.1}$$

on $\mathbf{C} \cup \{\infty\}$, with the usual conventions like $1/0 = \infty$, and where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is an invertible 2×2 matrix.